

Comment on the temperature dependence of the Casimir force.

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Linear in temperature correction to the Casimir force is discussed. The correction is important for small separations between bodies tested in the recent experiments and disappears in the case of perfect conductors.

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The Casimir force [1] has been measured with high precision in recent experiments [2–4]. Also there are plans [5,6] to look for very weak hypothetical forces where the Casimir force is the main background. All this makes the precise evaluation of the Casimir force an important problem. Here we will discuss a particular problem concerning the temperature dependence of the force between macroscopic bodies made of nonideal metals.

For perfect conductors the temperature correction has been found many years ago [7–9] and it is small for small separations between bodies $a \ll \hbar c/kT$ or equivalently for low temperature. For a sphere above a disk the leading term behaves as $(T/T_{eff})^3$, where $T_{eff} = \hbar c/2a$. This result follows from a general expression for the Casimir force given by Lifshitz [10,11] modified for the case of sphere-disk geometry with the proximity force theorem [12]:

$$F(a) = -\frac{kTR}{c^2} \sum_{n=0}^{\infty} \zeta_n^2 \int_1^{\infty} dp p \ln \left[\left(1 - G_1 e^{-2p\zeta_n a/c}\right) \left(1 - G_2 e^{-2p\zeta_n a/c}\right) \right], \quad (1)$$

where R is the sphere radius,

$$G_1 = \left(\frac{p-s}{p+s} \right)^2, \quad G_2 = \left(\frac{\varepsilon(i\zeta_n)p-s}{\varepsilon(i\zeta_n)p+s} \right)^2, \\ s = \sqrt{\varepsilon(i\zeta_n) - 1 + p^2}, \quad \zeta_n = \frac{2\pi n kT}{\hbar}, \quad (2)$$

$\varepsilon(i\zeta_n)$ is the dielectric function of the used material at imaginary frequencies. The prime over the sum sign indicates that the first term $n=0$ has to be taken with the coefficient $1/2$.

For small temperature the sum in (1) can be replaced by the integral over ζ and the resulting force does not depend on the temperature at all. In general, the replacement is true with the precision $\sim T/T_{eff}$. In condition of the atomic force microscope experiments [3,4] the smallest separation was $0.1 \mu m$ and the replacement error can be as large as 3%. It exceeds the experimental errors $\sim 1\%$ and, therefore, the finite temperature effect has to be taken into account. We define the temperature correction $\Delta_T F$ as difference between forces written as the sum over n and as the integral instead of this sum.

Special care needs to treat the first term $n=0$ in Eq.(1). The formal reason is that ζ_n^2 becomes zero but the integral over p diverges. The physical reason is that this term corresponds to the static limit when for metallic bodies $\varepsilon \rightarrow \infty$. This means that any parameter characterizing the dielectric function of a metal cannot appear in the $n=0$ term in contrast with a dielectric for which it will depend on the static permittivity of the material. In the $\varepsilon \rightarrow \infty$ limit the functions $G_{1,2}$ become $G_1 = G_2 = 1$. The formal problem is overcome by introducing the integration over a new variable $x = 2p\zeta_n a/c$ and after that one can take $\zeta_n = 0$ for the $n=0$ term. Transformed in this way Eq.(1) will be

$$F(a) = \frac{kTR}{4a^2} \left\{ \zeta(3) - \sum_{n=1}^{\infty} \int_{x_n}^{\infty} dx x \ln \left[(1 - G_1 e^{-x}) (1 - G_2 e^{-x}) \right] \right\}, \quad (3)$$

where $\zeta(m)$ is the zeta-function and

$$x_n = \frac{2\zeta_n a}{c}. \quad (4)$$

Here the first term is linear in temperature and it corresponds to the $n=0$ term in (1).

The sum in (3) as a function of temperature contains a piece linear in T which exactly cancels for ideal metals the first term giving the well known result

$$F_T(a) = F_0(a) \left[1 + \frac{45\zeta(3)}{\pi^3} \left(\frac{T}{T_{eff}} \right)^3 - \left(\frac{T}{T_{eff}} \right)^4 \right], \quad (5)$$

where $F_0(a) = \pi^3 \hbar c R / (360 a^3)$ is the bare Casimir force between sphere and plate. (5) is written in the small temperature limit when corrections to $F_0(a)$ are very small.

If we are using the dielectric function of a real metal, the cancellation of the first term in (3) can be incomplete and the linear in T contribution can survive. That was noted first in [13], where Eq.(3) was used for numerical calculation of the Casimir force. It was found that for the experiments [3,4] the temperature correction at the smallest separation is 4 pN against the experimental errors 2 pN . This conclusion has been criticized in Ref. [14], where the linear correction was not found. In this connection we would like to clarify here difference in the approaches.

The $n = 0$ term was discussed in [14] on the right basis but for actual calculations the following expression has been used

$$F(a) = -\frac{kTR}{4a^2} \sum_{n=0}^{\infty} \int_{x_n}^{\infty} dx \ln [(1 - G_1 e^{-x}) (1 - G_2 e^{-x})], \quad (6)$$

where for $n = 0$ the function $G_1 \neq 1$. It is clear from the expression for the force in the high temperature limit, where only the $n = 0$ term survives (Eq.(16) in [14])

$$F(a) = \frac{kT}{4a^2} R \zeta(3) \left(1 - \frac{2c}{a\omega_p} \right). \quad (7)$$

Here ω_p is the plasma frequency of the used metal. The parameter ω_p in this equation shows that the special prescription for the $n = 0$ term has not been done. The dielectric function was described by the plasma model where it is

$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta^2}. \quad (8)$$

In the high temperature limit only low frequency fluctuations are important and in this range metals can be much better described by the Drude dielectric function

$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta(\zeta + \omega_\tau)}, \quad (9)$$

where ω_τ is the relaxation frequency. However, if we use (9) to find the classical limit with the help of (6), the result will be wrong, namely, two times smaller than the well known limit $kTR\zeta(3)/4a^2$. Eq.(3) does not suffer from this problem.

The authors [14] convincingly demonstrated that for low temperatures Eq.(6) does not give the correction linear in T and the leading correction is only $(T/T_{eff})^3$. One can use this result to extract the linear term from the sum in Eq.(3) explicitly. The difference between (3) and (6) gives the correction we are looking for if one neglects the higher order terms in T/T_{eff}

$$\Delta_T F = \frac{kTR}{4a^2} \left\{ \zeta(3) + \frac{1}{2} \int_0^\infty dx \ln [(1 - G_1 e^{-x}) (1 - G_2 e^{-x})] \right\}. \quad (10)$$

The integral here is the linear term contained in the sum in (3) and, of course, it can depend on the material parameters because the summation is going over nonzero frequencies ζ_n . On the other hand, since this integral appeared as the $n = 0$ term in (6), we should take the functions $G_{1,2}$ at $x_n = 0$. In this limit $G_2 = 1$ but $G_1 \neq 1$. Using then the relation $\int_0^\infty dx \ln(1 - e^{-x}) = -\zeta(3)$ one finds the final expression for the correction linear in T :

$$\Delta_T F = \frac{kTR}{8a^2} \left[\zeta(3) + \int_0^\infty dx \ln(1 - G_1 e^{-x}) \right], \quad (11)$$

where

$$G_1 = \left(\frac{x - \sqrt{x^2 + \alpha^{-2}}}{x + \sqrt{x^2 + \alpha^{-2}}} \right)^2, \quad \alpha = \frac{c}{2a\omega_p}.$$

Let us stress that (11) is true only for the plasma model. When $\omega_p \rightarrow \infty$ the correction disappears as it should be. Expansion in powers of α gives

$$\Delta_T F = \frac{kTR}{8a^2} \zeta(3) \cdot 8\alpha (1 - 3\alpha + O(\alpha^2)). \quad (12)$$

For $\omega_p = 2 \cdot 10^{16} \text{ s}^{-1}$ and $a = 0.1 \text{ } \mu\text{m}$ one gets $\Delta_T F \approx 2.5 \text{ pN}$ using (11) or calculating directly with the help of (3) and 2.9 pN using (12). The correction increases further if we will use the Drude dielectric function (9). In this case it has to be evaluated numerically using (3) and (1) with the integral instead of the sum. The relaxation frequency ω_τ influences mostly on the integral since it changes low frequency behavior of the integrand. For typical value $\omega_\tau = 5 \cdot 10^{13} \text{ s}^{-1}$ we found $\Delta_T F \approx 4.0 \text{ pN}$. It cannot be compared directly with the value given in [13] because layered body cover has been considered there but it is clear that the calculations here give the same order of magnitude for $\Delta_T F$.

In conclusion, we have considered the linear in temperature correction to the Casimir force at low temperatures or equivalently at small separations. Special care has to be taken to get the contribution of the fluctuations in the static limit ($n = 0$ term). This contribution is canceled for ideal mirrors but cancellation is incomplete for real metals. The right treatment of the $n = 0$ term allowed to use the Drude dielectric function for metals which is more appropriate at low frequencies than the function in the plasma model.

- [1] H.B.G. Casimir, Koninkl. Ned. Adak. Wetenschap. Proc. **51**, 793 (1948).
- [2] S.K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997); **81**, 5475 (1998).
- [3] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998). physics/9805032
- [4] A. Roy, C. -Y. Lin, and U. Mohideen, Phys. Rev. **D 60**, 111101 (1999). quant-ph/9906062
- [5] J.C. Long, H.W. Chan, and J.C. Price, Nucl. Phys. **B 539**, 23 (1999). hep-ph/9805217
- [6] E. Fischbach and C. Talmadge, *The Search for Non-Newtonian Gravity* (AIP Press/Springer-Verlag, New York, 1999).
- [7] J. Mehra, Physica **37**, 145 (1967).
- [8] L.S. Brown and G.J. Maclay, Phys. Rev. **184**, 1272 (1969).
- [9] J. Schwinger, L.L. DeRaad, Jr., and K.A. Milton, Ann. Phys. (N.Y.) **115**, 1 (1978).
- [10] E.M. Lifshitz, Sov. Phys. JETP **2**, 73 (1956).
- [11] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2*, (Pergamon Press, Oxford, 1980).
- [12] J. Blocki, J. Randrup, W.J. Swiatecki, and C.F. Tsang, Ann. Phys. (N.Y.) **105**, 427 (1977).
- [13] V.B. Svetovoy and M.V. Lokhanin, e-print quant-ph/0001010.
- [14] M. Bordag, B. Geyer, G.L. Klimchitskaya, and V.M. Mostepanenko, e-print quant-ph/0003021.